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Quantum inverse scattering method for a non-linear N-wave resonance interaction system with both boson and fermion fields

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Abstract. The non-linear N-wave resonance interaction system with both boson and fermion fields is studied in the framework of the quantum inverse scattering method. The model Hamiltonian is diagonalised and the degeneracy of the eigenstates and the existence of the quantum bound states are analysed. Moreover, the classical limit is also discussed.

1. Introduction

Recently Kulish [1] studied the quantum non-linear three-wave resonance interaction model in the framework of the quantum inverse scattering method (QISM) [2-4]. The same model, but with a different choice of statistics proposed by Ohkuma and Wadati [5], has also been carried out by Wang and Pu [6]. On the other hand, the generalisation to the non-linear N-wave interaction system has been discussed by Kulish and Reshetikhin [7] for the algebraic Bethe ansatz equations on a finite interval of length L under periodic boundary conditions and by Zhou and Jiang [8] for the whole line ($L \to \infty$) in the case of a finite number of excitations. In this paper we discuss the quantum integrability of the non-linear N-wave interaction system with both boson and fermion fields. Our model may be viewed as a direct generalisation of that proposed by Ohkuma and Wadati [5]. Here we wish to stress that all integrable cases for the non-linear N-wave interaction system have been contained in our treatment.

The paper is organised as follows. In § 2 we present the model and construct the commutation relations for the quantum scattering data operators by solving the so-called Yang-Baxter relations. In § 3 the eigenstates for an infinite number of conservation laws in the model is constructued, and the degeneracy of the eigenstates and the existence of the quantum bound states are analysed. In § 4 the classical Yang-Baxter relations in the graded sense are presented, and then the classical limit is discussed. Our conclusions are summarised in § 5.

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2. Commutation relations

Our model is given by the Hamiltonian

$$\mathscr{H}=\int\left(\sum_{j<k} v_{jk}\frac{w_{jk}^{+}}{i}\frac{\partial w_{jk}}{\partial x}+\sum_{i<j<k}\varepsilon_{ijk}(w_{ik}^{+}w_{ij}w_{jk}+w_{jk}^{+}w_{ij}^{+}w_{ik})\right)\mathrm{d}x. \tag{1}$$

Here $w_{ij}^{+}(x)$ and $w_{ij}(x)$ are, respectively, creation and annihilation operators for boson or fermion fields, dependent on the corresponding parities being even or odd. Here and in the following, we always assign the parity of the field $w_{ij}(x)$ to be $P(i)+P(j)$, and set $P(i)=0$ ($1\leqslant i<\alpha$ or $\beta\leqslant i\leqslant N$) and $P(i)=1$ ($\alpha\leqslant i<\beta$, $\alpha,\beta=1,2,\ldots,N$). Then, $w_{ij}(x)$ and $w_{ij}^{+}(x)$ satisfy the usual equal-time (anti-)commutation relations

$$\begin{aligned}
&w_{ij}(x)w_{kl}(y)-(-1)^{[P(i)+P(j)][P(k)+P(l)]}w_{kl}(y)w_{ij}(x)=0\\
&\qquad w_{ij}^{+}(x)w_{kl}^{+}(y)-(-1)^{[P(i)+P(j)][P(k)+P(l)]}w_{kl}^{+}(y)w_{ij}^{+}(x)=0\\
&w_{ij}(x)w_{kl}^{+}(y)-(-1)^{[P(i)+P(j)][P(k)+P(l)]}w_{kl}^{+}(y)w_{ij}(x)=\delta_{ik}\delta_{jl}\delta(x-y)\\
&\qquad i<j;\ k<l;\ i,j,k,l=1,2,\ldots,N.
\end{aligned} \tag{2}$$

As shown in the appendix, the equations of motion of the system can be cast into the Lax form. For our purpose, let us consider the auxiliary linear problem in QISM in the form

$$\frac{\partial}{\partial x}\mathscr{T}(x,x_0|\lambda)=:\mathscr{L}(x,\lambda)\mathscr{T}(x,x_0|\lambda):\qquad \mathscr{T}(x_0,x_0|\lambda)=I \tag{3}$$

with

$$\mathscr{L}(x,\lambda)=\mathrm{i}\lambda\sum_{l}a_l e_{ll}+\sum_{l<m}l_{lm}(w_{lm}e_{lm}+w_{lm}^{+}e_{ml}) \tag{4}$$

where λ is the spectral parameter and e_{lm} is an $N\times N$ matrix with elements $(e_{lm})_{ij}=\delta_{il}\delta_{jm}$. In terms of parameters from (4), the group velocities v_{ij} and the coupling constants ε_{ijk} in (1) can be expressed as

$$v_{ij}=\frac{b_i-b_j}{a_i-a_j}\qquad \varepsilon_{ijk}=\frac{(-1)^{P(k)}l_{ij}l_{ik}l_{jk}}{\mathrm{i}c} \tag{5}$$

with

$$l_{ij}^2=(-1)^{P(j)}c\beta_{ij}\qquad \beta_{ij}=a_i-a_j=v_{ik}-v_{jk}. \tag{6}$$

In order to obtain the commutation relations for the quantum scattering data operators, we rewrite (3) in a lattice form:

$$\mathscr{T}_{j+1}(\lambda)=:\mathscr{L}_j(\lambda)\mathscr{T}_j(\lambda):\qquad \mathscr{T}_j(\lambda)\equiv\mathscr{T}(x_j,x_0|\lambda) \tag{7}$$

with

$$\mathscr{L}_j(\lambda)=\sum_{l}(I+\mathrm{i}\lambda a_l\Delta)e_{ll}+\sum_{l<m}l_{lm}(w_{lmj}e_{lm}+w_{lmj}^{+}e_{ml}). \tag{8}$$

Here Δ is the small lattice spacing and $w_{lmj}=w_{lm}(x_j)\Delta$. In (8), we have neglected the terms of order Δ^2. Thus the Yang–Baxter relations take the form

$$\mathscr{R}(\lambda-\mu)(\mathscr{L}_j(\lambda)\underset{s}{\otimes}\mathscr{L}_j(\mu))=(\mathscr{L}_j(\mu)\underset{s}{\otimes}\mathscr{L}_j(\lambda))\mathscr{R}(\lambda-\mu). \tag{9}$$

Here by $\underset{s}{\otimes}$ we mean the Grassmann direct product [9], which is defined by

$$(\mathscr{A} \underset{s}{\otimes} \mathscr{B})_{ik,jl} = (-1)^{[P(i)+P(j)]P(k)} \mathscr{A}_{ij} \mathscr{B}_{kl}. \tag{10}$$

It is easy to verify that the $\mathscr{R}$ matrix is given by

$$\mathscr{R}(\lambda-\mu) = \frac{-\mathrm{i}c}{\lambda-\mu-\mathrm{i}c} \sum_{lm} e_{ll} \otimes e_{mm} + \frac{\lambda-\mu}{\lambda-\mu-\mathrm{i}c} \sum_{lm} (-1)^{P(l)P(m)} e_{lm} \otimes e_{ml}. \tag{11}$$

It is worthwhile noticing that the Hilbert space of quantum states of the system under consideration is the tensor product of $N(N-1)/2-(N-M)M$ Fock spaces for boson fields and $(N-M)M$ Fock spaces for fermion fields:

$$H = \bigotimes_{i<j} H_{ij} \qquad i,j = 1,2,\ldots,N$$

with the pseudovacuum being defined by $w_{ij}(x)|0\rangle = 0$, $i<j$, $i,j=1,2,\ldots,N$, and $M=\beta-\alpha$. Here $H_{ij}(1 \leqslant i < \alpha \leqslant j < \beta$ or $\alpha \leqslant i < \beta \leqslant j \leqslant N)$ denote the Fock spaces for fermion fields and the others for boson fields. Now we can calculate the expectation value of $\mathscr{L}_j(\lambda) \underset{s}{\otimes} \mathscr{L}_j(\mu)$ between the pseudovacuum $|0\rangle$:

$$W(\lambda,\mu) = \sum_{lm} (1+\mathrm{i}\lambda a_l \Delta + \mathrm{i}\mu a_m \Delta) e_{ll} \otimes e_{mm} + c\Delta \sum_{l<m} (-1)^{P(l)P(m)} \beta_{lm} e_{lm} \otimes e_{ml}. \tag{12}$$

When proceeding to the continuum limit, we introduce a normalised monodromy matrix which is determined by

$$\mathscr{T}(\lambda) = \lim_{N\to\infty} V^{-N}(\lambda) \mathscr{L}_N(\lambda) \cdots \mathscr{L}_{-N+1}(\lambda) V^{-N}(\lambda) \tag{13}$$

where

$$V(\lambda) = \sum_l (1+\mathrm{i}\lambda a_l \Delta) e_{ll}. \tag{14}$$

Then the Yang-Baxter relations for $\mathscr{T}(\lambda)$ become

$$\mathscr{R}_+(\lambda-\mu)(\mathscr{T}(\lambda) \underset{s}{\otimes} \mathscr{T}(\mu)) = (\mathscr{T}(\mu) \underset{s}{\otimes} \mathscr{T}(\lambda)) \mathscr{R}_-(\lambda-\mu) \tag{15}$$

where

$$\mathscr{R}_+(\lambda-\mu) = U_+^{-1}(\mu,\lambda) \frac{\mathscr{R}(\lambda-\mu)}{\lambda-\mu} U_+(\lambda,\mu)$$

$$\mathscr{R}_-(\lambda-\mu) = U_-(\mu,\lambda) \frac{\mathscr{R}(\lambda-\mu)}{\lambda-\mu} U_-^{-1}(\lambda,\mu) \tag{16}$$

with

$$U_+(\lambda,\mu) = \lim_{N\to\infty} W^{-N}(\lambda,\mu)(V^N(\lambda) \otimes V^N(\mu))$$

$$U_-(\lambda,\mu) = \lim_{N\to\infty} (V^N(\lambda) \otimes V^N(\mu)) W^{-N}(\lambda,\mu). \tag{17}$$

After a very tedious but straightforward algebraic calculation, we obtain

$$\mathcal{R}_{\pm}(\lambda-\mu)=\sum_l\left(\frac{1}{\lambda-\mu}+\frac{(-1)^{P(l)}-1}{\lambda-\mu-\mathrm{i}c}\right)e_{ll}\otimes e_{ll}\pm\mathrm{i}\pi\delta(\lambda-\mu)\sum_{lm}\eta_{lm}e_{ll}\otimes e_{mm}$$
$$+\frac{\lambda-\mu+\mathrm{i}c}{(\lambda-\mu+\mathrm{i}\varepsilon)^2}\sum_{l<m}(-1)^{P(l)P(m)}e_{lm}\otimes e_{ml}$$
$$+\frac{1}{\lambda-\mu-\mathrm{i}c}\sum_{l>m}(-1)^{P(l)P(m)}e_{lm}\otimes e_{ml} \tag{18}$$

with

$$\eta_{lm}=\mathrm{sgn}(a_l-a_m)\qquad l,m=1,2,\ldots,N.$$

Here the δ function appears as a result of the formula

$$\lim_{L\to\infty}\frac{e^{\mathrm{i}Lx}}{x}=\mathrm{i}\pi\delta(x).$$

From this we immediately get the commutation relations for the scattering data operators. For later uses, we write out some useful relations below:

$$[\mathcal{A}_l(\lambda),\mathcal{A}_m(\mu)]=0\qquad l,m=1,2,\ldots,N$$
$$\mathcal{B}_{lm}(\lambda)\mathcal{B}_{lm}(\mu)=\frac{(-1)^{P(l)}(\lambda-\mu)+\mathrm{i}c}{(-1)^{P(m)}(\lambda-\mu)+\mathrm{i}c}\mathcal{B}_{lm}(\mu)\mathcal{B}_{lm}(\lambda)$$
$$\mathcal{C}_{lm}(\lambda)\mathcal{C}_{lm}(\mu)=\frac{(-1)^{P(m)}(\lambda-\mu)+\mathrm{i}c}{(-1)^{P(l)}(\lambda-\mu)+\mathrm{i}c}\mathcal{C}_{lm}(\mu)\mathcal{C}_{lm}(\lambda)$$
$$\mathcal{B}_{lm}(\lambda)\mathcal{A}_l(\mu)=\frac{\lambda-\mu+(-1)^{P(l)}\mathrm{i}c}{\lambda-\mu+\mathrm{i}\eta_{lm}\varepsilon}\mathcal{A}_l(\mu)\mathcal{B}_{lm}(\lambda)$$
$$\mathcal{B}_{lm}(\lambda)\mathcal{A}_m(\mu)=\frac{\lambda-\mu-(-1)^{P(m)}\mathrm{i}c}{\lambda-\mu-\mathrm{i}\eta_{lm}\varepsilon}\mathcal{A}_m(\mu)\mathcal{B}_{lm}(\lambda)$$
$$\mathcal{A}_l(\lambda)\mathcal{C}_{lm}(\mu)=\frac{\lambda-\mu-(-1)^{P(l)}\mathrm{i}c}{\lambda-\mu-\mathrm{i}\eta_{lm}\varepsilon}\mathcal{C}_{lm}(\mu)\mathcal{A}_l(\lambda)$$
$$\mathcal{A}_m(\lambda)\mathcal{C}_{lm}(\mu)=\frac{\lambda-\mu+(-1)^{P(m)}\mathrm{i}c}{\lambda-\mu+\mathrm{i}\eta_{lm}\varepsilon}\mathcal{C}_{lm}(\mu)\mathcal{A}_m(\lambda)$$
$$[\mathcal{B}_{mn}(\lambda),\mathcal{A}_l(\mu)]=(-1)^{P(l)}(\eta_{ln}-\eta_{lm})\pi c\delta(\lambda-\mu)\mathcal{C}_{lm}(\lambda)\mathcal{B}_{ln}(\lambda) \tag{19}$$
$$[\mathcal{B}_{lm}(\lambda),\mathcal{A}_n(\mu)]=(-1)^{P(l)P(m)+[P(l)+P(m)]P(n)}(\eta_{ln}-\eta_{mn})\pi c\delta(\lambda-\mu)\mathcal{C}_{mn}(\lambda)\mathcal{B}_{ln}(\lambda)$$
$$[\mathcal{A}_l(\lambda),\mathcal{C}_{mn}(\mu)]=(-1)^{P(l)}(\eta_{ln}-\eta_{lm})\pi c\delta(\lambda-\mu)\mathcal{C}_{ln}(\lambda)\mathcal{B}_{lm}(\lambda)$$
$$[\mathcal{A}_n(\lambda),\mathcal{C}_{lm}(\mu)]=(-1)^{P(l)P(m)+[P(l)+P(m)]P(n)}(\eta_{ln}-\eta_{mn})\pi c\delta(\lambda-\mu)\mathcal{C}_{ln}(\lambda)\mathcal{B}_{mn}(\lambda)$$
$$\mathcal{B}_{ln}(\lambda)\mathcal{A}_m(\mu)=\frac{(\lambda-\mu+\mathrm{i}c)(\lambda-\mu-\mathrm{i}c)}{(\lambda-\mu+\mathrm{i}\varepsilon)^2}\mathcal{A}_m(\mu)\mathcal{B}_{ln}(\lambda)$$
$$+(-1)^{P(m)}(\eta_{lm}+\eta_{mn})\pi c\delta(\lambda-\mu)\mathcal{B}_{lm}(\lambda)\mathcal{B}_{mn}(\lambda)$$
$$\mathcal{A}_m(\lambda)\mathcal{C}_{ln}(\mu)=\frac{(\lambda-\mu+\mathrm{i}c)(\lambda-\mu-\mathrm{i}c)}{(\lambda-\mu+\mathrm{i}\varepsilon)^2}\mathcal{C}_{ln}(\mu)\mathcal{A}_m(\lambda)$$
$$+(-1)^{P(m)}(\eta_{lm}+\eta_{mn})\pi c\delta(\lambda-\mu)\mathcal{C}_{mn}(\lambda)\mathcal{C}_{lm}(\lambda)\qquad l<m<n.$$

Here we have used the notation

$$\mathcal{T}(\lambda)=\sum_{l}\mathcal{A}_l(\lambda)e_{ll}+\sum_{l<m}(\mathcal{B}_{lm}(\lambda)e_{lm}+\mathcal{C}_{lm}(\lambda)e_{ml}). \tag{20}$$

3. Conserved quantities, eigenstates and bound states

Let us now discuss the scattering states for the model. From the Neumann series for $\mathcal{A}_l$ and $\mathcal{B}_{lm}$, we have

$$\mathcal{A}_l|0\rangle=|0\rangle \qquad \mathcal{B}_{lm}|0\rangle=0 \qquad l<m;\ l,m=1,2,\ldots,N. \tag{21}$$

The commutation relations between $\mathcal{A}_l$ and $\mathcal{C}_{mn}$ show that the state

$$\prod_{l<m}\prod_{k=1}^{n_{lm}}\mathcal{C}_{lm}(\lambda_k^{(lm)})|0\rangle \tag{22}$$

is an eigenstate of all $\mathcal{A}_l(\lambda)$ and the corresponding eigenvalues $a_l(\lambda)$ are determined by

$$\begin{aligned}a_l(\lambda)=&\prod_{i=1}^{l-1}\prod_{k=1}^{n_{il}}\frac{\lambda-\lambda_k^{(il)}+(-1)^{P(l)}\mathrm{i}c}{\lambda-\lambda_k^{(il)}}\prod_{i<l<j}\prod_{k=1}^{n_{ij}}\frac{(\lambda-\lambda_k^{(ij)}+\mathrm{i}c)(\lambda-\lambda_k^{(ij)}-\mathrm{i}c)}{(\lambda-\lambda_k^{(ij)})^2}\\&\times\prod_{i=l+1}^{N}\prod_{k=1}^{n_{li}}\frac{\lambda-\lambda_k^{(li)}-(-1)^{P(l)}\mathrm{i}c}{\lambda-\lambda_k^{(li)}}.\end{aligned} \tag{23}$$

On the other hand, $\mathcal{A}_l(\lambda)$ are the generating functionals for an infinite number of conservation laws in the model. Note that, when λ is very large, $\mathcal{A}_l(\lambda)$ have the following asymptotic expansions:

$$\mathcal{A}_l(\lambda)=1+\frac{\tau_l}{\lambda}+\frac{\mu_l}{\lambda^2}+\cdots \tag{24}$$

where

$$\tau_l=(-1)^{P(l)}\mathrm{i}c\mathcal{N}_l \tag{25}$$

$$\begin{aligned}\mu_l=&(-1)^{P(l)}\mathrm{i}c\left(\sum_{i=1}^{l-1}\frac{\mathcal{P}_{il}}{\beta_{il}}-\sum_{i=l+1}^{N}\frac{\mathcal{P}_{li}}{\beta_{li}}\right)+(-1)^{P(l)}\left(-\sum_{i<j<l}\frac{(-1)^{P(l)}l_{il}l_{ij}l_{jl}}{\beta_{il}\beta_{jl}}\mathcal{V}_{ijl}\right.\\&\left.+\sum_{i<l<j}\frac{(-1)^{P(j)}l_{il}l_{ij}l_{lj}}{\beta_{il}\beta_{lj}}\mathcal{V}_{ilj}-\sum_{l<i<j}\frac{(-1)^{P(j)}l_{li}l_{ij}l_{lj}}{\beta_{li}\beta_{lj}}\mathcal{V}_{lij}\right)\\&+\tfrac{1}{2}\tau_l^2+(-1)^{P(l)}\frac{\mathrm{i}c}{2}\tau_l+\mathrm{i}c\sum_{i=1}^{l-1}(-1)^{P(i)}\tau_i\end{aligned} \tag{26}$$

with

$$\begin{aligned}&\mathcal{N}_l=\int\left(\sum_{i=1}^{l-1}w_{il}^{+}w_{il}-\sum_{i=l+1}^{N}w_{li}^{+}w_{li}\right)\mathrm{d}x \qquad \mathcal{P}_{ij}=\int\frac{w_{ij}^{+}}{i}\frac{\partial w_{ij}}{\partial x}\mathrm{d}x\\&\mathcal{V}_{ijk}=\int(w_{ik}^{+}w_{ij}w_{jk}+w_{jk}^{+}w_{ij}^{+}w_{ik})\mathrm{d}x.\end{aligned} \tag{27}$$

From this we can reconstruct the well known conserved quantities, i.e. charges, momentum and Hamiltonian:

$$\mathcal{N}_l=\frac{(-1)^{P(l)}}{\mathrm{i}c}\tau_l \qquad \mathcal{P}=\frac{1}{\mathrm{i}c}\sum_{l=2}^{N}(-1)^{P(l)}\beta_{1l}\left(\mu_l-\tfrac{1}{2}\tau_l^2-(-1)^{P(l)}\frac{\mathrm{i}c}{2}\tau_l-\mathrm{i}c\sum_{i=1}^{l-1}(-1)^{P(i)}\tau_i\right)$$
$$\mathcal{H}=\frac{1}{\mathrm{i}c}\sum_{l=2}^{N}(-1)^{P(l)}v_{1l}\beta_{1l}(\mu_l-\tfrac{1}{2}\tau_l^2-(-1)^{P(l)}\frac{\mathrm{i}c}{2}\tau_l-\mathrm{i}c\sum_{i=1}^{l-1}(-1)^{P(i)}\tau_i\Big). \tag{28}$$

Expanding the eigenvalues $a_l(\lambda)$ for $\mathcal{A}_l(\lambda)$ in terms of the inverse powers of λ, we have

$$a_l(\lambda)=1+\frac{t_1}{\lambda}+\frac{u_l}{\lambda^2}+\cdots \qquad t_l=(-1)^{P(l)}\mathrm{i}c\left(\sum_{i=1}^{l-1}n_{il}-\sum_{i=l+1}^{N}n_{li}\right)$$
$$u_l=(-1)^{P(l)}\mathrm{i}c\left(\sum_{i=1}^{l-1}\sum_{k=1}^{n_{il}}\lambda_k^{(il)}-\sum_{i=l+1}^{N}\sum_{k=1}^{n_{li}}\lambda_k^{(li)}\right)+\tfrac{1}{2}t_l^2+(-1)^{P(l)}\frac{\mathrm{i}c}{2}t_l+\mathrm{i}c\sum_{i=1}^{l-1}(-1)^{P(i)}t_i. \tag{29}$$

Comparing this with (24) and (28), we immediately get

$$N_l=\sum_{i=1}^{l-1}n_{il}-\sum_{i=l+1}^{N}n_{li} \qquad P=\sum_{i<j}\beta_{ij}\sum_{k=1}^{n_{ij}}\lambda_k^{(ij)} \qquad E=\sum_{i<j}v_{ij}\beta_{ij}\sum_{k=1}^{n_{ij}}\lambda_k^{(ij)}. \tag{30}$$

Thus, the state (22) is the common eigenstate for an infinite number of conservation laws in the model.

To conclude this section, let us now discuss the degeneracy of the eigenstates and the existence of the quantum bound states. From (19), it follows that the states $\mathscr{C}_{l,l+2}(\lambda)|0\rangle$ and $\mathscr{C}_{l,l+1}(\lambda)\mathscr{C}_{l+1,l+2}(\lambda)|0\rangle$ have the same eigenvalue for all the conserved quantities. Further, the same statement also holds for the states $\mathscr{C}_{l,l+3}(\lambda)|0\rangle$, $\mathscr{C}_{l,l+1}(\lambda)\mathscr{C}_{l+1,l+3}(\lambda)|0\rangle$, $\mathscr{C}_{l,l+2}(\lambda)\mathscr{C}_{l+2,l+3}(\lambda)|0\rangle$ and $\mathscr{C}_{l,l+1}(\lambda)\mathscr{C}_{l+1,l+2}(\lambda)\mathscr{C}_{l+2,l+3}(\lambda)|0\rangle$, and so on. Generally, we can show that the states $\mathscr{C}_{ik}(\lambda)|0\rangle$ and $\mathscr{C}_{ij}(\lambda)\mathscr{C}_{jk}(\lambda)|0\rangle$ are degenerate. This reflects the fact that the system (1) describes the decay resonance interaction represented by

$$w_{ij}+w_{jk}\Leftrightarrow w_{ik}.$$

As for the existence of the quantum bound states, let us restrict ourselves to the case in which the fields $w_{jN}(j=1,2,\ldots,N-1)$ are regarded as fermions. In this case, we find that

$$\mathscr{C}_{jN}(\lambda+\mathrm{i}c)\mathscr{C}_{jN}(\lambda)=0 \qquad j=1,2,\ldots,N-1. \tag{31}$$

This implies that the fields w_{jN} $(j=1,2,\ldots,N-1)$ cannot make bound states. On the other hand, the bound state of $m^{(l,l+1)}w_{l,l+1}$ particles occurs when the corresponding spectral parameters form a string:

$$\lambda_k^{(l,l+1)}=\lambda^{(l,l+1)}-\mathrm{i}c(m^{(l,l+1)}+1-2k)/2$$
$$k=1,2,\ldots,m^{(l,l+1)};\ \mathrm{Im}\,\lambda^{(l,l+1)}=0;\ l=1,2,\ldots,N-2 \tag{32}$$

and when the group velocities satisfy

$$(v_{l,l+1}-v_{l+1,k})(v_{l,l+1}-v_{lk})>0 \qquad k=l+2,\ldots,N \tag{33}$$

respectively. Indeed, if we put the system on a finite interval of length L, then the periodic boundary condition leads to

$$\exp(\mathrm{i}\beta_{l,l+1}\lambda_\gamma L)=-\prod_{\delta=1}^{m}\frac{\lambda_\gamma-\lambda_\delta+\mathrm{i}c}{\lambda_\gamma-\lambda_\delta-\mathrm{i}c} \qquad \gamma,\delta=1,2,\ldots,m. \tag{34}$$

Here we have dropped the superscripts for convenience. In particular, in the two-particle case, we have

$$\exp(\mathrm{i}\beta_{1,1+1}\lambda_1 L)=\frac{\lambda_1-\lambda_2+\mathrm{i}c}{\lambda_1-\lambda_2-\mathrm{i}c}. \tag{35}$$

Obviously, the condition for the complex parameters λ to exist is $\lambda_1^*=\lambda_2$. Setting $\lambda_1=\lambda+\mathrm{i}\kappa$ and $\lambda_2=\lambda-\mathrm{i}\kappa$ $(\kappa>0)$, and substituting into (35), we find that, when L goes to infinity, (35) requires $\beta_{l,l+1}>0$ if $c=-2\kappa<0$ or $\beta_{l,l+1}<0$ if $c=2\kappa>0$. This implies that $c\beta_{l,l+1}<0$. Note that when $\varepsilon_{ijk}^2=-c\beta_{ij}\beta_{ik}\beta_{jk}$ and $\beta_{ij}=v_{ik}-v_{jk}$ we immediately get the conditions (33). It is interesting to note that the binding energy of the bound state is zero. This must be a consequence of the fact that the system is linear dispersive. When the conditions (33) are satisfied for all l, the bound states of the fields $w_{l,l+1}(l=1,2,\ldots,N-2)$ coexist. If so, the bound states involving other fields also occur. This is obvious if we notice the degeneracy of the eigenstates discussed above. No doubt, it is interesting to compare our results with those of Ohkuma and Wadati [5]. Unfortunately, it is difficult to rederive the Bethe eigenvectors constructed by them from the quantum inverse scattering method, although this can be done by using a generalisation of Wiesler's method for the non-linear Schrödinger model [10]. We shall return to this problem elsewhere along with a detailed study of the algebraic Bethe ansatz equations for the model (1).

4. Classical limit

Let us now come back to the classical theory. Then, the classical equations of motion of the system are determined by

$$\dot{w}_{ij}=\{w_{ij},\mathscr{H}\}. \tag{36}$$

Here the Poisson bracket is defined by

$$\{a,b\}=\mathrm{i}\int\sum_{i<j}\left(\frac{\delta a}{\delta w_{ij}}\frac{\delta b}{\delta w_{ij}^*}-(-1)^{P(i)+P(j)}\frac{\delta a}{\delta w_{ij}^*}\frac{\delta b}{\delta w_{ij}}\right)\mathrm{d}x \tag{37}$$

where the functional derivatives are understood in the sense of Grassman algebras [11]. As in the non-linear Schrödinger model, the classical $\mathscr{L}$ matrix takes the same form as (5), and is then a supermatrix. Thus, the classical Yang–Baxter relations should also be understood in the graded sense:

$$\{\mathscr{L}(x,\lambda)\underset{s}{\otimes}\mathscr{L}(y,\mu)\}=[r(\lambda-\mu),\mathscr{L}(x,\lambda)\underset{s}{\otimes}I+I\underset{s}{\otimes}\mathscr{L}(x,\mu)]\delta(x-y) \tag{38}$$

or, in a lattice form,

$$\{\mathscr{L}_i(\lambda)\underset{s}{\otimes}\mathscr{L}_j(\mu)\}=[r(\lambda-\mu),\mathscr{L}_i(\lambda)\underset{s}{\otimes}I+I\underset{s}{\otimes}\mathscr{L}_i(\mu)]\delta_{ij}. \tag{39}$$

Here the Poisson brackets of Grassman direct product of two supermatrices $\mathscr{A}$ and $\mathscr{B}$ are defined by

$$\{\mathscr{A}\underset{s}{\otimes}\mathscr{B}\}=\mathrm{i}\int\sum_{i<j}\left(\frac{\delta\mathscr{A}}{\delta w_{ij}}\underset{s}{\otimes}\frac{\delta\mathscr{B}}{\delta w_{ij}^*}-(-1)^{P(i)+P(j)}\frac{\delta\mathscr{A}}{\delta w_{ij}^*}\underset{s}{\otimes}\frac{\delta\mathscr{B}}{\delta w_{ij}}\right)\mathrm{d}x \tag{40}$$

or, equivalently,

$$\{\mathscr{A} \underset{s}{\otimes} \mathscr{B}\}_{ik,jl} = (-1)^{[P(i)+P(j)]P(k)}\{\mathscr{A}_{ij}, \mathscr{B}_{kl}\}. \tag{41}$$

In our case, the r matrix is

$$r(\lambda-\mu) = -\frac{c}{\lambda-\mu}\sum_{lm}(-1)^{P(l)P(m)} e_{lm}\otimes e_{ml}. \tag{42}$$

It is easy to check that the quasiclassical correspondence given by Izergin and Korepin [12]

$$\mathscr{R} \xrightarrow[\hbar\to 0]{} \sum_{lm}(-1)^{P(l)P(m)} e_{lm}\otimes e_{ml}\left(\sum_{lm} e_{ll}\otimes e_{mm} - \mathrm{i}\hbar r\right) \tag{43}$$

also holds in our case. The corresponding classical Yang-Baxter relations for the normalised monodromy matrix take the form

$$\{\mathscr{T}(\lambda) \underset{s}{\otimes} \mathscr{T}(\mu)\} = r_+(\lambda-\mu)\mathscr{T}(\lambda) \underset{s}{\otimes} \mathscr{T}(\mu) - \mathscr{T}(\lambda) \underset{s}{\otimes} \mathscr{T}(\mu) r_-(\lambda-\mu) \tag{44}$$

where

$$r_\pm(\lambda-\mu) = \lim_{x\to+\infty} E(-x,\lambda)\otimes E(-x,\mu) r(\lambda-\mu) E(x,\lambda)\otimes E(x,\mu) \tag{45}$$

with

$$E_{ij}(x,\lambda) = \exp(\mathrm{i}\lambda a_i x)\delta_{ij}. \tag{46}$$

Then, we have

$$r_\pm(\lambda-\mu) = -\frac{c}{\lambda-\mu}\sum_{l}(-1)^{P(l)} e_{ll}\otimes e_{ll} \pm \mathrm{i}\pi c\delta(\lambda-\mu)\sum_{lm}(-1)^{P(l)P(m)}\eta_{lm} e_{lm}\otimes e_{ml}. \tag{47}$$

Thus we can write out some classical Poisson brackets for the scattering data:

$$\begin{aligned}
&\{\mathscr{A}_l(\lambda), \mathscr{A}_m(\mu)\} = 0 \qquad l, m = 1, 2, \ldots, N \\
&\{\mathscr{B}_{lm}(\lambda), \mathscr{B}_{lm}(\mu)\} = [(-1)^{P(m)} - (-1)^{P(l)}]\frac{c}{\lambda-\mu}\mathscr{B}_{lm}(\lambda)\mathscr{B}_{lm}(\mu) \\
&\{\mathscr{C}_{lm}(\lambda), \mathscr{C}_{lm}(\mu)\} = [(-1)^{P(l)} - (-1)^{P(m)}]\frac{c}{\lambda-\mu}\mathscr{C}_{lm}(\lambda)\mathscr{C}_{lm}(\mu) \\
&\{\mathscr{B}_{lm}(\lambda), \mathscr{A}_l(\mu)\} = \frac{-(-1)^{P(l)}c}{\lambda-\mu+\mathrm{i}\eta_{lm}\varepsilon}\mathscr{B}_{lm}(\lambda)\mathscr{A}_l(\mu) \\
&\{\mathscr{B}_{lm}(\lambda), \mathscr{A}_m(\mu)\} = \frac{(-1)^{P(m)}c}{\lambda-\mu-\mathrm{i}\eta_{lm}\varepsilon}\mathscr{B}_{lm}(\lambda)\mathscr{A}_m(\mu) \\
&\{\mathscr{A}_l(\lambda), \mathscr{C}_{lm}(\mu)\} = \frac{(-1)^{P(l)}c}{\lambda-\mu-\mathrm{i}\eta_{lm}\varepsilon}\mathscr{A}_l(\lambda)\mathscr{C}_{lm}(\mu) \\
&\{\mathscr{A}_m(\lambda), \mathscr{C}_{lm}(\mu)\} = \frac{-(-1)^{P(m)}c}{\lambda-\mu+\mathrm{i}\eta_{lm}\varepsilon}\mathscr{A}_m(\lambda)\mathscr{C}_{lm}(\mu) \\
&\{\mathscr{B}_{mn}(\lambda), \mathscr{A}_l(\mu)\} = (-1)^{P(l)}(\eta_{ln} - \eta_{lm})\mathrm{i}\pi c\delta(\lambda-\mu)\mathscr{C}_{lm}(\lambda)\mathscr{B}_{ln}(\lambda)
\end{aligned} \tag{48}$$

$$\{\mathscr{B}_{lm}(\lambda), \mathscr{A}_n(\mu)\} = (-1)^{P(n)}(\eta_{ln} - \eta_{mn})\mathrm{i}\pi c\delta(\lambda-\mu)\mathscr{B}_{ln}(\lambda)\mathscr{C}_{mn}(\lambda)$$

$$\{\mathscr{A}_l(\lambda), \mathscr{C}_{mn}(\mu)\} = (-1)^{P(l)}(\eta_{ln} - \eta_{lm})\mathrm{i}\pi c\delta(\lambda-\mu)\mathscr{C}_{ln}(\lambda)\mathscr{B}_{lm}(\lambda)$$

$$\{\mathscr{A}_n(\lambda), \mathscr{C}_{lm}(\mu)\} = (-1)^{P(n)}(\eta_{ln} - \eta_{mn})\mathrm{i}\pi c\delta(\lambda-\mu)\mathscr{B}_{mn}(\lambda)\mathscr{C}_{ln}(\lambda)$$

$$\{\mathscr{B}_{ln}(\lambda), \mathscr{A}_m(\mu)\} = (-1)^{P(m)}(\eta_{lm} + \eta_{mn})\mathrm{i}\pi c\delta(\lambda-\mu)\mathscr{B}_{lm}(\lambda)\mathscr{B}_{mn}(\lambda)$$

$$\{\mathscr{A}_m(\lambda), \mathscr{C}_{ln}(\mu)\} = (-1)^{P(m)}(\eta_{lm} + \eta_{mn})\mathrm{i}\pi c\delta(\lambda-\mu)\mathscr{C}_{mn}(\lambda)\mathscr{C}_{lm}(\lambda) \qquad l<m<n.$$

From this we can see that the correspondence between quantum commutation relations and classical Poisson brackets follows the usual principle:

$$\{a, b\} = \mathrm{i}[a, b] \tag{49}$$

5. Conclusions

We have studied the non-linear N-wave resonance interaction system with both boson and fermion fields in the framework of the quantum inverse scattering method. Our model may be viewed as a direct generalisation of that proposed by Ohkuma and Wadati [5]. In fact, if we regard the fields $w_{1\alpha}, \ldots, w_{\alpha-1,\alpha}, w_{\alpha,\alpha+1}, \ldots, w_{\alpha N}$ as fermions for a fixed α, and note that the exchange between w_{ij} and $w_{N+1-j,N+1-i}$ does not affect the physics of the system, then $N/2$ $((N+1)/2)$ different choices of statistics are only independent for even (odd) N, respectively. Thus, there are three different choices in the three-wave interaction model.

We have determined the energy spectrum of the quantum Hamiltonian for the model, and analysed the existence of the quantum bound states. In the classical limit, we have presented the classical Yang–Baxter relations in the graded sense and examined the quasiclassical correspondence previously found by Izergin and Korepin [12]. We wish that our formulation may also be useful in other completely integrable systems. A possible candidate is a generalised non-linear Schrödinger model associated with the superalgebra osp$(m, 2n)$, which may be considered as a reduction of the system (1).

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Appendix. Derivation of the Lax pair for system (1)

The traditional basis for applying the inverse scattering method is to represent the equations of motion of the system in the Lax form:

$$\frac{\partial}{\partial t}\mathscr{T}(x, y|\lambda) =: \mathscr{M}(x, \lambda)\mathscr{T}(x, y|\lambda): \qquad \frac{\partial}{\partial x}\mathscr{T}(x, y|\lambda) =: \mathscr{L}(x, \lambda)\mathscr{T}(x, y|\lambda): \tag{A1}$$

where $\mathscr{L}$ and $\mathscr{M}$ are $N \times N$ matrices depending on the spectral parameter λ and the dynamical variables. The compatibility condition of (A1)

$$\mathscr{L}_t - \mathscr{M}_x + [\mathscr{L}, \mathscr{M}] = 0 \tag{A2}$$

should be consistent with the equations of motion of the system. In our case, we assume

$$\mathscr{L}(x, \lambda) = \mathrm{i}\lambda \sum_l a_l e_{ll} + \sum_{lm} p_{lm}(x) e_{lm} \tag{A3}$$

$$\mathscr{M}(x, \lambda) = -\mathrm{i}\lambda \sum_{lm} b_{lm} e_{lm} + \sum_{lm} q_{lm}(x) e_{lm}.$$

Substituting (A3) into (A2), we have

$$(a_l b_{lm} - b_{lm} a_m)\lambda^2 + \left(a_l q_{lm} - q_{lm} a_m + \sum_{n=1}^{N} (b_{ln} p_{nm} - p_{ln} b_{nm}) \right) \mathrm{i}\lambda$$

$$+ \frac{\partial p_{lm}}{\partial t} - \frac{\partial q_{lm}}{\partial x} + \sum_{n=1}^{N} (p_{ln} q_{nm} - q_{ln} p_{nm}) = 0. \tag{A4}$$

Obviously, the coefficients in the same powers of λ in the above equation must be zero. Therefore we have

$$a_l b_{lm} - b_{lm} a_m = 0 \tag{A5}$$

$$a_l q_{lm} - q_{lm} a_m + \sum_{n=1}^{N} (b_{ln} p_{nm} - p_{ln} b_{nm}) = 0 \tag{A6}$$

and

$$\frac{\partial p_{lm}}{\partial t} - \frac{\partial q_{lm}}{\partial x} + \sum_{n=1}^{N} (p_{ln} q_{nm} - q_{ln} p_{nm}) = 0. \tag{A7}$$

From this we conclude

$$\begin{aligned} & b_{lm} = b_l \delta_{lm} \\ & q_{lm} = -v_{lm} p_{lm} \\ & \frac{\partial p_{lm}}{\partial t} + v_{lm} \frac{\partial p_{lm}}{\partial x} = -\sum_{n=1}^{N} (v_{ln} - v_{nm}) p_{ln} p_{nm}. \end{aligned} \tag{A8}$$

Here we have used the notation

$$v_{lm} = \frac{b_l - b_m}{a_l - a_m}.$$

Now choosing

$$P_{lm} = \begin{cases} l_{lm} w_{lm} & l < m \\ 0 & l = m \\ l_{ml} w_{ml}^{+} & l > m \end{cases} \tag{A9}$$

where

$$l_{lm}^2 = (-1)^{P(m)} c \beta_{lm} \qquad \beta_{lm} = a_l - a_m = v_{ln} - v_{mn}$$

and substituting (A9) into (A8), we have

$$q_{lm} = \begin{cases} -v_{lm} l_{lm} w_{lm} & l < m \\ 0 & l = m \\ -v_{lm} l_{ml} w^{+}_{ml} & l > m \end{cases} \tag{A10}$$

and

$$\begin{aligned}\frac{\partial w_{lm}}{\partial t} + v_{lm}\frac{\partial w_{lm}}{\partial x} &= -(-1)^{P(m)}\frac{l_{lm}}{c}\left(\sum_{n=1}^{l-1} l_{nl} l_{nm} w^{+}_{nl} w_{nm} + \sum_{m=l+1}^{m-1} l_{ln} l_{nm} w_{ln} w_{nm}\right. \\ &\quad \left. + \sum_{n=m+1}^{N} l_{ln} l_{mn} w_{ln} w^{+}_{mn}\right).\end{aligned} \tag{A11}$$

Comparing the above equation with the equations of motion, we obtain

$$\varepsilon_{ijk} = \frac{(-1)^{P(k)} l_{ij} l_{jk} l_{ik}}{\mathrm{i} c}. \tag{A12}$$

It must be noticed that there are always appropriate choices to guarantee the reality of the coupling constants ε_{ijk}.

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